

Lower Bound of Electrical Conductivity from Holography

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Abstract

We propose a universal lower bound of dc electrical conductivity in rotational- and translational- symmetries breaking systems via the holographic duality. This bound predicts that BTZ-black brane can be easily used to realize linear temperature resistivity. We also construct an anisotropic black brane solution, which yields linear temperature for the in-plane resistivity and insulating behavior for the out-of-plane resistivity. Ultimately, we discuss its implications in experiments.

Introduction

The AdS/CFT correspondence provides a powerful tool to analyze strongly coupled systems, particularly for studying the transport properties of strongly coupled systems. One of the most famous results of the AdS/CFT applications, is the so-called Kovtun-Son-Starinets (KSS) bound $\eta/s \geq \hbar/(4\pi k_B)$, which states that for strongly coupled systems with a classical Einstein gravity dual description, the ratio of the shear viscosity η , to the entropy density s , obeys such a bound [1]. For higher derivative gravity, the bound is violated by a numerical factor, but the KSS bound can still be stated with a specific constant factor C , with $\eta/s \geq C\hbar/k_B$ [2–6].

Recently, the KSS conjecture was severely challenged by the anisotropic black brane systems, where the shear viscosity is a tensor and some components of the tensor can become considerably smaller, which parametrically violates the bound [7, 8]. Considered a $d + 1$ -dimensional geometry with coordinates (t, x_i, z) , and anisotropy only along the z -direction, the shear viscosity to entropy density ratio is related to the anisotropy as follows

$$\frac{\eta_{x_i z, x_i z}}{s} = \frac{\hbar}{4\pi k_B} \frac{g_{x_i x_i}}{g_{zz}} \Big|_{r=r_H}, \quad (1)$$

where $g_{x_i x_i}$ and g_{zz} are the line elements of the metric and r_H is the event horizon radius, respectively. An arbitrary violation of the KSS bound would occur if $g_{x_i x_i}/g_{zz} \rightarrow 0$. In this anisotropic background, the rotational symmetry of the dual field theory is broken from $SO(d-1)$ to $SO(d-2)$. We thus have shear viscosities $\eta_{x_i z, x_i z}$, which are defined by the metric fluctuations $h_{x_i z}$. Such metric components carry spin 1 with respect to the $SO(d-2)$ symmetries [9]. Although the spin-2 components of the shear viscosity tensor in the $x_i - x_j$ plane satisfy the KSS bound, the shear force in the $x_i - z$ plane, which is related to the spin-1 metric components, can violate it. Furthermore, the diffusion bound $D \gtrsim C\hbar v_F^2/k_B T$ will also break down [10–14]. The diffusivity bound was proposed to replace the Mott-Ioffe-Regel (MIR) bound in bad metals, and it is based on the KSS bound $\eta/s \geq C\hbar/k_B$ and the relation $\eta/s = DT/c^2$ for a vanishing chemical potential [11].

One natural question is whether there is an alternative bound to be obeyed by the transport coefficients in such anisotropic systems. It is well-known that in condensed matter physics, it is notably universal that the materials are anisotropic with different properties in different directions. Remarkably, the transport in high- T_c cuprates is strongly two-dimensional in character and there is substantial anisotropy between the in- and out-of-plane (i.e. CuO_2 plane) resistivities. In contrast to the resistivity ρ_{ab} in the CuO_2 planes,

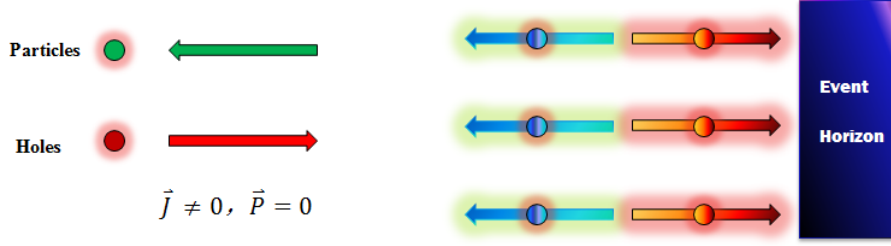


Figure 1: The particle-hole-like creation leading to charge current in one-dimensional Luttinger liquid is analogous to pair production of the Hawking radiation leading to BTZ black hole dissipation. On the dual field theory side, quantum critical charge current without momentum arises from thermally excited particle-hole-like pairs. Maximal possible relaxation rate is set by temperature alone [15].

where a generic behavior is observed to depend on the metallic temperature, the c -axis transport in high-temperature cuprates is very highly material-specific. Intriguingly, in most underdoped cuprates, $\rho_c(T)$ shows insulating behavior at all temperatures [16].

Therefore, the universal transporting properties of anisotropic systems deserve further studies. In this study, we will show that the ratio of the determinant of the electrical DC conductivities to the graviton absorption cross-section in anisotropic systems from holography in the zero-charges limit has a universal value

$$\left. \frac{\prod_i \sigma_{ii}}{\mathcal{A}^{d-3}} \right|_{q_i=0} = Z_H^{d-1}, \quad (2)$$

where \mathcal{A} , q_i and Z_H are area density per unit volume of the black hole event horizon, the electric charges and gauge field coupling, respectively. In the minimal coupling case, $Z_H = 1$. This bound should also hold for the isotropic systems. One justification of the above ansatz is that the vector Goldstone modes are generated because of the $SO(d-1)$ to $SO(d-2)$ symmetry breaking; the components of the shear viscosity, which leads to the viscosity bound violation that corresponds to metric perturbation with spin-1, should act similarly to the conductivity [9]. It is natural to expect that the conductivity behavior may be more universal than the shear viscosities in such rotational-symmetry-breaking systems.

There is a universal relation between the graviton absorption cross-section and the black-brane horizon area in the large-incident-wavelength limit [17]: $\mathcal{A} = \Sigma(\omega = 0)$. The rise of the event horizon area-graviton cross-section equivalence is simply because the metric perturbation component $h_{x_i}^{x_j}$ satisfies the equation of motion of the minimally coupled massless scalar $\square h_{x_i}^{x_j} = 0$. The spin-2 shear viscosity component is proportional to

the graviton absorption cross-section via $\eta_{x_i x_j, x_i x_j} = \Sigma(\omega = 0)/2\kappa^2$. Therefore, the spin-2 shear viscosity component is linearly dependent on the event horizon area (i.e. the entropy density). However, for the spin-1 component $h_z^{x_i}$, the equation of motion is not identical to minimally coupled massless Klein-Gordon equation and thus the absorption cross-section of spin-1 vector field $h_{x_i z}$ in an anisotropic black-brane background is not equal to the black-brane horizon area.

The universal relation (2) is able to provide us some insights into the holographic realizations of the linear temperature resistivity as shown in figure 1. Considering the simplest case with minimal gauge coupling $Z(\phi) = 1$ and isotropic transporting behavior, one knows that the local thermodynamic stability of black branes in AdS space requires $c_q = T(\partial s/\partial T)_q > 0$ and the entropy density s is proportional to \mathcal{A} . Thus, we can make some predictions based on equation (2):

- 1). Linear-T resistivity can be easily obtained in 2 + 1-dimensional charged Bañados-Teitelboim-Zanelli (BTZ) black hole background. This black hole is dual to one-dimensional Luttinger liquid [18, 19].
- 2). For $Z(\phi) = 1$ and $d \geq 3$, isotropic black branes in the AdS space cannot be utilized to realize linear temperature resistivity in the zero-charges limit. Nevertheless, anisotropic black branes are good candidates in model-building of holographic strange metals.
- 3). For $d + 1$ -dimensional spatially isotropic Lifshitz black holes, this bound indicates that $\sigma_{ii}|_{q_i=0} = [4\pi/(d+z-1)]^{d-3} T^{(d-3)/z}$, which is consistent with what obtained in Refs.[20, 21] based on a universal scaling relation hypothesis: $\sigma(\omega = 0) = T^{(d-3)/z} \Theta(0)$, where z is a dynamical critical exponent and $\Theta(\omega)$ is a frequency dependent function.

Basic setup

Without loss of generality, we consider the Einstein-Maxwell-dilaton action with linear scalar fields

$$S = \int d^{d+1}x \sqrt{-g} \left[\frac{1}{16\pi G} \left(R - \frac{1}{2} \partial\phi^2 + V(\phi) - \frac{1}{2} \sum_{i=1}^{p-1} Y_i(\phi) \partial\psi_i^2 \right) - \frac{1}{4g_{d+1}^2} Z(\phi) F^2 \right]. \quad (3)$$

Hereafter, we select $16\pi G = g_{d+1}^2 = L = 1$, where L is the AdS radius, g_{d+1}^2 is the $d + 1$ -dimensional gauge coupling constant, and G is Newton's constant. Recently, this model has been widely studied in Refs. [22–29]. The solution to the above theory is assumed to be

anisotropic

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{xx}dx^jdx^j + g_{zz}dzdz, \quad (4)$$

$$\phi = \phi(r), \quad A = A_t(r)dt, \quad \psi_j = k_jx_j, j = 1 \cdots d-3, \quad \psi_z = k_zz, \quad k_j \neq k_z.$$

The anisotropic direction is selected along the z -direction. We regard the $x_i - x_j$ plane as the “ ab ” plane and the z -direction as the “ c ”-axis in cuprates. We assume that there is a regular event horizon at $r = r_H$ with the following ansatz [30]

$$g_{tt} \sim \frac{1}{g_{rr}} \sim 4\pi T(r - r_H), \quad (5)$$

$$\phi \sim \phi_H + \cdots, \quad A_t \sim a_H(r - r_H) + \cdots, \quad (6)$$

$$g_{xx} \sim g_{xx}(r_H) + \cdots, \quad g_{zz} \sim g_{zz}(r_H) + \cdots \quad (7)$$

The entropy density is given by $s = 4\pi(g_{xx}^{d-2}g_{zz})^{1/2}|_{r=r_H}$. The electric charge density is given by $q \equiv -J^t = -\sqrt{-g}Z(\phi)\partial_r A_t$. Near the boundary, the metric ansatz is assumed to have asymptotic forms

$$g_{tt} \sim \frac{1}{g_{rr}} \sim r^2, \quad g_{xx} \sim g_{zz} \sim r^2, \quad (8)$$

$$A_t \sim \mu + qr^{-d+2} + \cdots, \quad \phi \sim \lambda r^{\Delta-d} + \cdots \quad (9)$$

where $\Delta = \frac{1}{2}[(d-1) \pm \sqrt{(d-1)^2 + 4m^2}]$ is the scaling dimension of the dual scalar field.

DC electric conductivities along isotropic directions with momentum dissipation

We impose a constant electric field in the x_i direction with magnitude E , which will generate electric currents only along the x_j direction. Let us consider a small perturbation in the black hole background

$$A_j = -Et + \delta a_{x_j}(r), \quad g_{tx_j} = \delta g_{tx_j}(r), \quad g_{rx_j} = g_{xx}\delta h_{rx_j}(r), \quad \psi = k_jx_j + \delta\chi_1. \quad (10)$$

From Maxwell equation $\partial_r(\sqrt{-g}Z(\phi)F^{rx_i}) = 0$, we can define a conserved current $J^{x_j} = -\sqrt{-g}g^{rr}g^{xx}Z(\phi)\partial_r a_{x_j} + \delta g_{tx_j}g^{xx}q$. In the absence of a charge density, we only have a contribution to the current from the gauge field $J^{x_j} \sim \partial_r a_{x_j}$. The conductivity can be determined based on the horizon regularity. In this case, we simply have

$$\left(\sqrt{\frac{g_{tt}}{g_{rr}}}a'_{x_j}\right)' = 0. \quad (11)$$

Regularity at the horizon gives us

$$a_{x_j} = -\frac{E}{4\pi T} \ln(r - r_H). \quad (12)$$

At finite charge density, we must know the behavior of δg_{tx_j} at the horizon. In the presence of momentum dissipation, δg_{tx_j} will take a finite value at the horizon

$$\delta g_{tx_j} = \frac{Eq}{k_j^2 Y_H g_{xx}^{\frac{d-3}{2}}} \Big|_{r=r_H}, \quad (13)$$

where we use the notation $Y_H = Y(\phi_H)$ and $Z_H = Z(\phi_H)$. Therefore, the conserved current is obtained as

$$J^{x_j} = \left(g_{xx}^{\frac{d}{2}-2} g_{zz}^{\frac{1}{2}} Z_H E + \frac{Eq^2}{k_j^2 Y_H g_{xx}^{\frac{d-1}{2}}} \right) \Big|_{r=r_H}. \quad (14)$$

Then, the DC conductivity is given by

$$\sigma_{jj} = \frac{J^{x_j}}{E} = \left(g_{xx}^{\frac{d}{2}-2} g_{zz}^{\frac{1}{2}} Z_H + \frac{q^2}{k_j^2 Y_H g_{xx}^{\frac{d-1}{2}}} \right) \Big|_{r=r_H}. \quad (15)$$

The first term in the above formula corresponds to the conductivity of the particle-hole pair creation and the second term is associated with the momentum relaxation. In the following section, we will calculate the DC conductivity along the anisotropic z -direction.

Conductivity anisotropy

Conductivity anisotropy is important in condensed matter physics, because the divergence of the resistivity anisotropy, and the temperature-linear resistivity at optimal doping of the cuprates are among the most puzzled problems to theorists. Normal-state transport in high- T_c cuprates has become one of the most challenging topics in condensed matter physics. A clear understanding of the normal-state transport properties of cuprates is considered as a key step towards understanding the pairing mechanism for high-temperature superconductivity.

To calculate the DC conductivity along the anisotropic direction, we must impose a constant electric field in the z direction with magnitude E_z . Now, we consider a small perturbation along the z -direction

$$A_z = -E_z t + \delta a_z(r), \quad g_{tz} = \delta g_{tz}(r), \quad g_{rx_i} = g_{zz} \delta h_{rz}(r), \quad \psi = k_z z + \delta \chi_z. \quad (16)$$

In this case, the conserved current is given by

$$J^z = -\sqrt{-g} g^{rr} g^{zz} Z(\phi) \partial_r a_z + \delta g_{tz} g^{zz} q. \quad (17)$$

Regularity at the horizon provides us the solution for a_z ,

$$a_z = -\frac{E_z}{4\pi T} \ln(r - r_H). \quad (18)$$

In the presence of momentum relaxation, δg_{tz} can be determined as follows:

$$\delta g_{tz} = \frac{Eq\sqrt{g_{zz}}}{k_z^2 Y_H g_{xx}^{\frac{d}{2}-1}} \Big|_{r=r_H}. \quad (19)$$

Then, the conserved current is obtained as

$$J^z = \left(g_{xx}^{\frac{d}{2}-1} g_{zz}^{-\frac{1}{2}} Z_H E_z + \frac{E_z q^2}{k_z^2 Y_H g_{xx}^{\frac{d}{2}-2} \sqrt{g_{zz}}} \right) \Big|_{r=r_H}. \quad (20)$$

The DC conductivity along the anisotropic direction is

$$\sigma_{zz} = \left(g_{xx}^{\frac{d}{2}-1} g_{zz}^{-\frac{1}{2}} Z_H + \frac{q^2}{k_z^2 Y_H g_{xx}^{\frac{d}{2}-2} \sqrt{g_{zz}}} \right) \Big|_{r=r_H}. \quad (21)$$

The ratio between the isotropic and anisotropic DC conductivities in the zero-charge limit $q \rightarrow 0$ can be easily evaluated

$$\frac{\sigma_{zz}}{\sigma_{jj}} = \frac{g_{xx}}{g_{zz}} \Big|_{r=r_H}. \quad (22)$$

For notably small $k_j \sim k_z \ll q$, the translation symmetry is weakly broken and it is expected to have a Drude peak of conductivity. The dissipation term becomes dominant when the particle-hole production term is negligible. The conductive anisotropy becomes

$$\frac{\sigma_{zz}}{\sigma_{jj}} \sim \frac{\sqrt{g_{xx}}}{\sqrt{g_{zz}}} \Big|_{r=r_H}. \quad (23)$$

A concrete calculation was provided in Ref.[31], where the DC conductivity along the z -direction exhibits an insulating behavior with $d\rho_{DC}/dT < 0$, which is consistent with the c -axis transport behavior for underdoped cuprates. The R-charged version of the anisotropic black brane solution was constructed using the non-linear Kaluza-Klein reduction of type-IIB supergravity [32, 33], where $g_{x_i x_i} = r_H^2 e^{-\phi_H/2}$ and $g_{zz} = r_H^2 e^{-3\phi_H/2}$. Therefore, we achieve the result $\sigma_{zz}/\sigma_{ii} = e^{-\phi_H} < 1$ since $\phi_H > 0$.

This result is qualitatively consistent with our observation in cuprates, where the resistive anisotropy ρ_c/ρ_{ab} varies between from 10 to over 10^6 at the critical temperature [16]. The ratio of shear viscosities also satisfies the above relation: $\eta_{x_i x_j, x_i x_j}/\eta_{x_i z, x_i z} = \sigma_{zz}/\sigma_{jj}$.

Linear-T Resistivity from Charged BTZ black holes

The simplest case of linear-T resistivity can be derived from the $2+1$ -dimensional charged BTZ black holes. Utilizing the action (3), we simply set

$$d = 2, \quad \phi = 0, \quad V(\phi) = 2, \quad Y(\phi) = 1, \quad Z(\phi) = 1. \quad (24)$$

One of the solutions to the above action is given by the 2 + 1-dimensional charged BTZ metric and a linear scalar field

$$ds^2 = \frac{1}{z^2} \left(-f(z)dt^2 + dx^2 + \frac{dz^2}{f(z)} \right), \quad (25)$$

$$f(z) = 1 - \frac{z^2}{z_h^2} + \frac{\mu^2 + c_1^2}{2} z^2 \ln \frac{z}{z_h}, \quad (26)$$

$$A_t(z) = \mu \ln \frac{z}{z_h}, \quad \psi_1 = c_1 x. \quad (27)$$

The black hole temperature and entropy density are given by $T = \frac{4-z_h^2(\mu^2+c_1^2)}{8\pi z_h}$ and $s = 4\pi z_h^{-1} \sim T$, respectively. In the higher temperature limit, the DC conductivity is obtained as

$$\sigma_{xx} = \frac{1}{T} + \frac{\mu^2}{c_1^2 T}. \quad (28)$$

Therefore, the linear-T resistivity appears both in the quantum critical term and momentum dissipation term. In the zero charge limit, we arrive at $\sigma_{QC} = 1/T$. Physics in 1 + 1-dimensions involve very interesting phenomena in condensed matter physics such as spin chains, quantum wires and Luttinger liquids. The first term in equation (28) is qualitatively consistent with the result obtained in Ref. [34].

Linear-T Resistivity from Anisotropic Black Holes

To demonstrate how the linear resistivity can be realized in higher dimensions, we consider an anisotropic systems with the following action:

$$S = \int d^5x \sqrt{-g} \left[R + 12\Lambda - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}e^{2\alpha\phi}(\partial\chi_3)^2 - \frac{1}{4}F^2 - \frac{1}{2}\sum_{i=1}^2(\partial\chi_i)^2 \right]. \quad (29)$$

In the low-temperature limit, the solution is given by

$$ds^2 = l^2 \left(-r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 dx^2 + r^2 dy^2 + r^{\frac{4\alpha^2}{1+2\alpha^2}} dz^2 \right), \quad (30)$$

$$\chi_i = \beta_{ia} x^a, \quad \chi_3 = c_3 z, \quad \phi = \frac{2\alpha}{1+2\alpha^2} \ln u, \quad c_3 = \frac{\sqrt{2(3+8\alpha^2)}}{1+2\alpha^2}, \quad l^2 = \frac{3+8\alpha^2}{4+8\alpha^2}. \quad (31)$$

It is easy to verify that the above ansatz yields a solution in the absence of charge

$$f(r) = 1 - \frac{\beta^2(1+2\alpha^2)}{(2+8\alpha^2)u^2} + \frac{\beta^2(1+2\alpha^2)}{(2+8\alpha^2)r^2} \left(\frac{r_H}{r} \right)^{\frac{l^2}{4}} - \left(\frac{r_H}{r} \right)^{\frac{l^2}{4}}, \quad (32)$$

$$\beta^2 = \frac{1}{2} \sum_1^2 \vec{\beta}_a \cdot \vec{\beta}_a, \quad \vec{\beta}_a \cdot \vec{\beta}_b = \beta^2 \delta_{ab}. \quad (33)$$

The Hawking temperature is given by $T = \frac{r_H l^2}{16\pi} - \frac{\beta^2(1+2\alpha^2)l^2}{(2+8\alpha^2)r_H}$ and the entropy density is proportional to the temperature. The quantum critical conductivity at high temperature is given by

$$\sigma_{jj} = r_H^{\frac{2\alpha^2}{1+2\alpha^2}}, \quad \sigma_{zz} = r_H^{2-\frac{2\alpha^2}{1+2\alpha^2}}. \quad (34)$$

For the special case of $\alpha^2 = -\frac{1}{4}$ and $\beta^2/r_H \rightarrow 0$ but $\beta^2/(1+4\alpha^2)$ is finite, we can easily obtain

$$\sigma_{jj} = \frac{16\pi}{l^2} T^{-1}, \quad \sigma_{zz} = \frac{4096\pi^3}{l^6} T^3. \quad (35)$$

The resistivity, i.e., σ_{jj}^{-1} , varies linearly with T , which provides a phenomenological account for the linear resistivity of strange metals. σ_{zz} monotonically decreases with decreasing temperature and behaves as T^3 . The metallic behavior in the “ab”-plane and insulating behavior along the “c”-axis indicate that this anisotropic model captures the key features of the high- T_c transport properties in the normal state. The experimental data of σ_c is proportional to T^3 for YBa₂Cu₃O_{6.95} [35], which is consistent with our result that the T^3 power law of the c-axis conductivity [36] is notably *universal* for cuprates.

Conductivity Bound

As argued in the previous section, for systems with rotational-symmetry breaking, vector Goldstone bosons are generated. Such Goldstone modes correspond to the unbroken Lorentz symmetry in the boundary theory. Correspondingly, a residual AdS symmetry is preserved in the bulk. From the Kaluza-Klein reduction, we know that the off-diagonal components of the metric, whose perturbations carry spin 1, induce gauge fields in the dimensionally reduced theory. Jain et al. argued that the conductivity of these gauge fields, is proportional to the spin-1 viscosity components η_{x_iz, x_iz} [9], which motivates us to conjecture that the dc electrical conductivities obeys a universal lower bound in anisotropic systems.

Considering the above facts, we propose that the determinant of the quantum critical conductivity matrix has a scaling relation with the black hole horizon area: $\prod_i \sigma_{ii}|_{q_i=0} = \mathcal{A}^{d-3} Z_H^{d-1}$. We further speculate a new conductivity bound, that is similar to the conjectured lower bound for the viscosity:

$$\frac{\prod_i \sigma_{ii}}{\mathcal{A}^{d-3}} \geq Z_H^{d-1}. \quad (36)$$

This bound should be valid for a wide class of systems with arbitrary spatial inhomogeneity and anisotropy. For the special case $d = 3$ and $Z_H = 1$, the bound reduces to $\prod_i \sigma_{ii} \geq 1$,

which is exactly the universal lower bound on the electric conductivity of the minimal coupled 4-dimensional Einstein-Maxwell theory in Ref.[37]. In order to recast (36) in a experimental testable manner, we consider a generalized spatially anisotropic black hole with following line-elements

$$ds^2 = -r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 \sum_{i=1}^p dx_i^2 + r^{\frac{2}{z}} \sum_{j=p+1}^{d-1} dy_j^2, \quad (37)$$

where $f(r)$ is the blacken factor with event horizon locates at $r = r_H$. The horizon area density $\mathcal{A} = r_H^{p+(d-1-p)/z} \sim T^{p+(d-1-p)/z}$, where we have assumed $r_H \sim T$. Therefore, the bound reduces to

$$\prod_i \sigma_{ii} \geq \mathcal{C} T^{(p+\frac{(d-1-p)}{z})(d-3)} Z_H^{d-1}, \quad (38)$$

where \mathcal{C} is a specific constant to be determined. Equation (38) is a generalized form of quantum critical conductivity and it can recover the quantum critical conductivity given in Refs.[20, 21] under the conditions $Z_H = 1$ and $p = 0$.

The DC conductivity can be written as the sum of an explicit charge-dependent term and a quantum critical term

$$\sigma_{ii}^{DC} = \sigma_{ii}^{QC} + \frac{q^2}{\varepsilon + p} \tau_{ii}^L, \quad (39)$$

where ε and p are the energy density and pressure, respectively, and τ_L is a time scale associated with the impurity/lattice. Therefore, this theory has a “universal” finite conductivity even without a net charge density. The quantum critical current that is carried by the particle-hole pairs of opposite momenta, controls the rate at which charge diffuses instead of the momentum relaxation. However, in an anisotropic system, the quantum critical conductivity σ_{QC} along different directions should not be identical. We provide a simple proof of (36) in the presence of charge, where the DC conductivity receives an additional Drude term. The relaxation time τ_L must be positive, as observed in the dispersion relation of the shear diffusion mode

$$\omega = -i \left(\tau_L^{-1} + \frac{\eta k^2}{\varepsilon + p} \right) + \dots \quad (40)$$

When $\tau_L < 0$, the metric- fluctuations quasinormal modes absorb the momentum, which makes the small perturbation of the state grow exponentially in time. It is straightforward to prove that $\prod_i \sigma_{ii}^{DC}(q_i \neq 0) - \prod_i \sigma_{ii}^{DC}(q_i = 0) > 0$.

One may naturally relate the conductivity bound with the diffusion bound via the Einstein relation $\sigma = D\chi$. The charge susceptibility of a black hole is given by [38]

$$\chi = \left[\int_{r_H}^{\infty} dr' \frac{g_{rr} g_{tt}}{\sqrt{-g}} \right]^{-1}. \quad (41)$$

We have $\chi \sim (d-2)r_{\text{H}}^{d-2}$ only if $g_{tt}g_{rr} = 1$ and $g_{xx} \sim g_{zz} \sim r^2$. Then, the ratio between $\prod_i \sigma_{ij}$ and the charge susceptibility in the zero-charge limit then obeys [10]

$$\frac{(\prod_i \sigma_{ii})^{\frac{1}{d-1}}}{\chi} = \frac{1}{d-2} \frac{1}{\sqrt{g_{xx}}} \Big|_{r=r_{\text{H}}} \geq \frac{\hbar v^2}{4\pi T} \frac{d}{d-2}, \quad (42)$$

where we use the notation $T = \frac{dr_{\text{H}}}{4\pi}$. The above ansatz does not work for Lifshitz dynamical scaling $t \rightarrow \lambda^\epsilon z$, $x \rightarrow \lambda x$. Even under the relativistic scaling $t \rightarrow \lambda z$, $x \rightarrow \lambda x$, if the spatial direction of the spacetime is highly anisotropic, i.e., $g_{xx} \sim r^2$, $g_{zz} \sim r^\alpha$, where α is an arbitrary positive constant, the diffusion bound is not respected. Nevertheless, the conductivity bound (36) is well respected by the Lifshitz black hole and anisotropic black brane solutions in Ref.[9].

Alternatively, if we consider the ratio between $\det \sigma_{ij}$ and the entropy density s , we obtain the ratio between the gauge coupling constant and Newton's constant $\prod_i \sigma_{ii}/s^{d-3} = (4G)^{d-3}/g_{d+1}^{2(d-1)}$. From the dual gravitational viewpoint, this ratio is related to a version of the “weak gravity conjecture” of Ref.[39, 40].

Discussion and Conclusion

In this study, we propose a lower bound of the DC electrical conductivity. Holography provides us a uniquely tractable method to study those strongly interacting systems without quasiparticles. In incoherent metals without a Drude peak, the transport is described by diffusive physics in terms of the diffusion of charge and energy instead of momentum diffusion. Thus, one can propose that the conductivity is a more universal physical quantity in such systems. In the absence of isotropy, different metric perturbations break up into components with different spin values. The shear viscosity in a rotationally invariant field theory is proportional to the graviton absorption via $\eta = \Sigma(0)/2\kappa^2$. The spin 2 metric perturbation component obeys the equation of motion of a minimally coupled massless scalar. Therefore, the absorption cross-section of a graviton is equivalent to that of a scalar field. A theorem on the scalar absorption cross-section states that in the larger- wavelength limit, $\mathcal{A} = \Sigma(0)$. So, the shear viscosity is proportional to the black hole entropy density because $s = \mathcal{A}/4G$. However, for rotational- symmetry -breaking systems, for a metric perturbation with spin 1, the equation of motion for $h_z^{x_i}$ cannot be written in the form of a minimal coupled massless scalar. Instead, its equation of motion can be recast in a similar form to the Maxwell equation: $\nabla_\mu f^{\mu\nu} + \nabla_\mu g_{xx} f^{\mu\nu}/g_{xx} = 0$. Remarkably, we achieve a simple model, where the linear temperature resistivity and insulating behavior are realized

in the isotropic plane and out-of-plane, respectively.

The dc electrical conductivity bound proposed here (i.e., (36) and (38)) can provide some insights on future model building of linear temperature resistivity in holographic theory and it is experimentally testable because it can recover the form in Ref. [21]. If the dual Maxwell theory must be minimally coupled, the presence of the linear temperature resistivity and the universality of the conductivity bound infers that anisotropy is a fundamental factor to be considered in normal-state of high temperature superconductors. We expect that our result is falsifiable in the future study.

Note added. While finalizing this work, we received the paper [41], where the authors considered the black hole entropy in Horndeski gravity and calculated η/s . The authors found that the black hole entropy in Horndeski gravity is not equal to one quarter of the area of the event horizon. There is an additional contribution over and above those that come from the standard Wald entropy formula. They also found that the viscosity/entropy ratio can violate the KSS bound for appropriate choices of the parameters. Fortunately, the conductivity bound provided here is well respected by their black hole solution.

Appendix

Shear viscosities

For the anisotropic fluid, the viscosity tensor η_{ijkl} yields two shear viscosities out of among five independent components. For the longitudinal mode $h_{xz}(t, u, y) = h_{xz}(u)e^{-i\omega t + ik_y y}$, we obtain the equation of motion for h_{xz}

$$\partial_u(\mathcal{N}^{uz}\partial_u h_z^x) - k_y^2 \mathcal{N}^{zy} h_z^x - \omega^2 \mathcal{N}^{tz} h_z^x = 0, \quad (43)$$

with the following notation:

$$\mathcal{N}^{\mu\nu} = \frac{1}{2\kappa^2} g_{xx} \sqrt{-g} g^{\mu\mu} g^{\nu\nu}. \quad (44)$$

In this case, the Green function is defined as $G_{xz,xz} = \frac{\mathcal{N}^{uz}\partial_u h_z^x}{h_z^x}$. Thus, the longitudinal shear viscosity is thus given by $\eta_{xz,xz} = -\frac{G_{xz,xz}}{i\omega}$. The equation of motion for h_{xz} can simply be recast as follows:

$$\partial_u \eta_{xz,xz} = i\omega \left(\frac{\eta^2}{\mathcal{N}^{uz}} + \mathcal{N}^{tz} \right) + \frac{i}{\omega} \mathcal{N}^{zy} k_y^2. \quad (45)$$

The horizon regularity requires

$$\eta_{xz,xz} = \sqrt{\mathcal{N}^{uz}\mathcal{N}^{tz}} \Big|_{u=u_H} = \frac{s}{4\pi} g_{xx} g^{zz} \Big|_{u=u_H}. \quad (46)$$

Therefore, we have a parametric form of η/s

$$\frac{\eta_{xz,xz}}{s} = \frac{1}{4\pi} g_{xx} g^{zz}. \quad (47)$$

Similarly, for the transverse shear viscosity, we have

$$\frac{\eta_{xy,xy}}{s} = \frac{1}{4\pi} g_{xx} g^{yy} = \frac{1}{4\pi}. \quad (48)$$

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